

0! and Related Topics

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Most texts with anything to say about 0! usually make some such declaration as,

0! $\equiv$ 1 by definition.

This has the mathematical stature of the bill passed unanimously in 1897 by the Indiana House of Representatives setting

$\pi \equiv 3.2$

The proper thing to say is,

0! = 1 (*The proof is beyond the scope of this presentation.*)

The proof *is* the main point, however, of this presentation. Let's begin with a modified Laplace transform called the Gamma Function [$\Gamma(n)$] written as

$$[1] \quad \Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx \quad n > 0$$

This is a nasty integral with no general closed form solution, but it is a valid function. Substituting $n+1$ for n in [1] the next order function is

$$[2] \quad \Gamma(n+1) = \int_0^{\infty} e^{-x} x^n dx \quad n+1 > 0$$

Integrating [2] by parts per $\int u dv = u \cdot v - \int v du$, choosing $u = x^n$, $v = e^{-x} dx$

$$[3] \quad \Gamma(n+1) = -x^n e^{-x} \Big|_0^{\infty} + n \int_0^{\infty} e^{-x} x^{n-1} dx \quad n+1 > 0 \text{ results.}$$

The first term is $-\infty^n \cdot e^{-\infty} + 0^n \cdot 1 = 0$, since of x^n and e^{-x} , the latter converges more rapidly than the former diverges. The second term is $n \cdot \Gamma(n)$ from [1], providing the recurrence relation between orders > 0 as

$$[4] \quad \Gamma(n+1) = n\Gamma(n) \quad n > 0 \text{ for the recurrence relationship.}$$

Evaluating $\Gamma(1)$ directly by setting $n = 0$ in [2], the result is

$$[5] \quad \Gamma(0+1) = \Gamma(1) = \int_0^{\infty} e^{-x} dx = -e^{-x} \Big|_0^{\infty} = 0 - (-1) = 1$$

One obtains from the recurrence relation [4]

$$\begin{aligned}\Gamma(2) &= 1 \cdot \Gamma(1) = 1 \cdot 1 = 1 \\ \Gamma(3) &= 2 \cdot \Gamma(2) = 2 \cdot 1 \cdot 1 = 2 \\ \Gamma(4) &= 3 \cdot \Gamma(3) = 3 \cdot 2 \cdot 1 \cdot 1 = 6 \text{ and}\end{aligned}$$

$$[6] \quad \Gamma(n+1) = n \cdot \Gamma(n) = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1 \equiv n!$$

when n is an integer. It is seen that the direct calculation for $0!$ has been done in [5] *without resort to arbitrary definition*, thus

$$[7] \quad \Gamma(0+1) = \Gamma(1) = 0! = 1 \quad \text{QED} = \text{Quod Erat Demonstrandum} \\ \neq \text{Quite Easily Derived}$$

Note that $0 \cdot \Gamma(0)$ need not be a concern, as the integral in [1] does not converge for $\Gamma(0) = (-1)!$, whence the $n > 0$ stipulation. I quote MIT Professor F.B.Hildebrand on factorials of negative integers, "As it approaches negative one, it loses its braveness, and retreats to confusion." Generalizing [6] as a function of a complex variable, singularities appear only at negative integer arguments, as might be expected from [4].

A rather indirect method can be used to obtain the interesting $(-1/2)! = \sqrt{\pi}$.

The very accurate Stirling approximation is given here without derivation as:

$$[8] \quad n! = \Gamma(n+1) \approx \sqrt{2\pi} n^{n+1/2} e^{-n} \quad (n \rightarrow \infty), \text{ which is less than 8\% off for } n \text{ as low as 1} \\ \text{and improves rapidly as } n \text{ becomes large. Accuracy is within 1\% for } n \geq 9.$$

0! and Related Topics Continued

1986 November 6

Here is the rather indirect method used to obtain $(-1/2)! = \sqrt{\pi}$. [1] is rewritten in the variable z as

$$[9] \quad \Gamma(n) = \int_0^{\infty} e^{-z} z^{n-1} dz \quad n > 0$$

Evaluating for $n=1/2$ [9] becomes

$$[10] \quad \Gamma(1/2) = \int_0^{\infty} e^{-z} z^{-1/2} dz$$

Duplicating [10] and changing variables to $z=x^2$ and $z=y^2$ gives

$$[11] \quad \Gamma(1/2) = 2 \int_0^{\infty} e^{-x^2} dx \quad \text{and} \quad \Gamma(1/2) = 2 \int_0^{\infty} e^{-y^2} dy$$

Multiplying the left and right sides together gives

$$[12] \quad \left[\Gamma\left(\frac{1}{2}\right) \right]^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$$

This volume integral under the surface $w = e^{-(x^2+y^2)}$ is in the first quadrant $x \geq 0, y \geq 0$ and is converted to polar coordinates as

$$[13] \quad \left[\Gamma\left(\frac{1}{2}\right) \right]^2 = 4 \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta = 4 \left. \frac{\pi}{2} \frac{e^{-r^2}}{-2} \right|_0^\infty = \pi$$

Taking the square root yields

$$[14] \quad \Gamma\left(\frac{1}{2}\right) = (-\frac{1}{2}!) = \sqrt{\pi} \quad \mathbf{QED}$$

Note that [11] and [14] yield the normalization value

$$[15] \quad \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} = \Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \quad \text{from recurrence [4] applied to [14]}$$

A more interesting than useful equation [in these days of modern computers] given here without derivation is

$$[16] \quad (2n)! = \frac{2^{2n}}{\sqrt{\pi}} n! (n - \frac{1}{2})!$$